**Module 2: Basic Concepts of Trigonometric Functions**

**III. The Unit Circle and Angle Conversion**

After completing this section, you should be able to:

* convert between degree and radian measures
* understand the relationship between arc length and an angle measured in radians
* find linear speed and angular speed, and apply the relationship between them

**A. Radian Measure**

In the discussion so far, angles have been measured in degrees. However, there is another system of measurement of angles, called radian measure, which proves to be especially convenient in scientific and mathematical settings.

The concept of a radian is related to the unit circle, the circle of radius 1 centered at the origin.

Since the circumference of a circle of radius π is 2π*r*, the circumference of the unit circle is 2π.

|  |  |
| --- | --- |
| Suppose you walk counterclockwise on the unit circle, starting at the point (1, 0) on the positive *x*-axis.  If you walk counterclockwise on the circle until you reach the positive *y*-axis, you will have traveled one-fourth of the circumference: a distance of 2π/4, or π/2. The path you have taken, the arc, corresponds to an angle of 90°. |  |
| If you walk along the circle from the point (1, 0) to the negative *x*-axis, you will have traveled a distance of π (half of the circumference) and your arc of length π corresponds to an angle of 180°. |  |
| Now suppose you begin at the same starting point and travel a distance of 1 counterclockwise along the unit circle.  Then, in radian measure, the corresponding angle is defined to be 1 radian.  If you travel along an arc of length 2 on the unit circle, the corresponding angle is 2 radians. |  |
| The arc length *s* on the unit circle corresponds to an angle of *s* radians. |  |

Traveling a distance π along the unit circle corresponds to the angle of π radians. Earlier, it was noted that a distance of π corresponds to the angle 180°. So, there is a simple relationship between radian measure and degree measure:

180° = π radians.  


These equations can be used to convert units of measurement from degrees to radians, and from radians to degrees.

**Example III.A.1:** Convert 300° to radians.

**Solution:**



**Example III.A.2:** Convert –5π/4 radians to degrees.

**Solution:**



The following table lists the degree measure and the corresponding radian measure for a set of frequently used angles. The four background colors correspond to the four quadrants. The circle diagram illustrates the correspondence between degree measure and radian measure.

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | **Degree Measure** | **Radian Measure** | | 0° | 0 | | 30° | π/6 | | 45° | π/4 | | 60° | π/3 | | 90° | π/2 | | 120° | 2π/3 | | 135° | 3π/4 | | 150° | 5π/6 | | 180° | π | | 210° | 7π/6 | | 225° | 5π/4 | | 240° | 4π/3 | | 270° | 3π/2 | | 300° | 5π/3 | | 315° | 7π/4 | | 330° | 11π/6 | |  |

Considering that 180 degrees equals π radians, 360 degrees must then equal 2π radians. Recall that angles whose degree measures differ by an integer multiple of 360 degrees are coterminal. Therefore, angles whose radian measures differ by an integer multiple of 2πradians are coterminal, as well. Reference angles in radian measure are determined in an analogous fashion to reference angles in degree measure.

|  |  |  |
| --- | --- | --- |
| **Calculation of the Measure of the Reference Angle** | | |
| Quadrant  of Angle *θ* | Measure of Reference Angle sigma in Degrees (assuming  is the angle coterminal with *θ*, having measure between 0° and 360°) | Measure of Reference Angle sigma in Radians (assuming is the angle coterminal with *θ*, having measure between 0 and 2π radians) |
| I | phi | phi |
| II | 180° – phi | π – |
| III | phi – 180° | phi– π |
| IV | 360° –phi | 2π –phi |

**Example III.A.3:** Find three angles that are coterminal with the angle of radian measure 15π/4 and determine the reference angle, also measured in radians.

**Solution:**

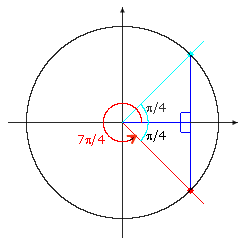
To find coterminal angles in radian measure, add or subtract multiples of 2π:

One coterminal angle has measure 15π/4 + 2π = 15π/4 + 8π/4 = 23π/4.  
Another coterminal angle has measure 15π/4 – 2π = 15π/4 – 8π/4 = 7π/4.  
Yet another coterminal angle has measure 15π/4 – 3(2π) = 15π/4 – 24π/4 = –9π/4.

There are infinitely many possible answers.

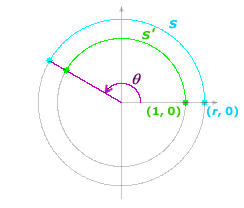
To find the reference angle, first find a coterminal angle whose measure is between 0 and 2π. The coterminal angle 7π/4 qualifies. Note that 7π/4 is a quadrant IV angle. The reference angle is 2π – 7π/4 = 8π/4 – 7π/4 = π/4.

A sketch of the associated reference triangle can be used to aid in finding the reference angle.



In the discussion so far, the focus has been placed on the unit circle. Similar ideas can be extended to any circle of radius *r* centered at the origin.

For angle *θ*, let *s* be the corresponding length of the arc on the circle of radius *r*.  
Let *s*' be the length of the corresponding arc on the unit circle.



The ratio of the arc lengths *s* and *s*' is the same as the ratio of the radius *r* and 1.

That is, .

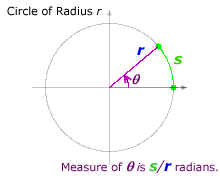
On the unit circle, the length of the arc *s*' is the measure of the angle *θ* in radians. In other words, *s*' = *θ*.

Therefore, ; equivalently, .

**Radian Measure**

For a circle of radius *r* with arc of length *s* and corresponding angle *θ*, *θ* = *s*/*r* in radian measure.

Equivalently, *s*= *rθ*, for *θ* measured in radians.



**Example III.A.4:** For a circle of radius 4 centimeters, find the length of the arc corresponding to an angle of 5π/6 radians.

**Solution:**  
*s* = *rθ* = 4(5π/6) = 10π/3 centimeters, approximately 10.47 centimeters.

From this point forward, when an angle measurement is stated and no units are given, it is assumed that the measurement is in radians.

**B. Linear Speed and Angular Speed**

When considering applied problems, the concept of speed often arises. For example, a car might travel a linear speed of 55 miles per hour. A tire might achieve an angular speed of 200 revolutions per minute, or 400π radians per minute.

The linear speed, *v*, is equal to the distance traveled, *s*, divided by the time elapsed, *t*: *v* = *s*/*t*.

The angular speed, traditionally denoted by a lowercase omega, or , is equal to the angle of rotation, *θ*, divided by the time elapsed, *t*:  = *θ*/*t*.

|  |  |
| --- | --- |
| Suppose you are sitting on a rotating disk, such as a merry-go-round. |  |

Regardless of where you sit on the disk, you will be traveling at the same angular speed, since all points on the disk will complete a 360° (or 2π radian) rotation in the same time. However, if you sit nearer the edge of the disk than the center, your linear speed will be greater. A point near the edge lies on a circle of greater radius than a point near the center. In one second, you will travel the same proportion of the circumference for either circle, but because the larger circle has a greater circumference, the distance traveled in that time will be greater, leading to a greater linear speed.

For angles measured in radians, there is a simple relationship between linear speed and angular speed. Suppose a point is located *r* units from the center of rotation and an arc of length *s* has been traversed in time *t*. Since *s* = *rθ*, linear speed = *v* = *s*/*t* = *rθ*/*t* = *r*, where is the angular speed, measured in radians per time unit.

Linear speed *v* and angular speed  are related by *v* = *r*.

To apply this relationship, be careful to make sure your units are consistent. The units of distance for *v* and *r* must be the same, the units of time for *v* and  must be the same.

**Example III.B.1:** If a merry-go-round of radius 12 feet makes one complete rotation in 15 seconds, what is the linear speed of an individual riding on the merry-go-round 2 feet from the edge?

**Solution:**

The angular speed is .

Since the radius of the merry-go-round is 12 feet, an individual 2 feet from the edge must be 10 feet from the center.

For a point on a circle of radius 10 feet, the linear speed, *v*, is calculated:



The individual is traveling at a linear speed of approximately 4.2 feet per second.

[*Return to top of page*](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/The%20Unit%20Circle%20and%20Angle%20Conversion.html#pagetop)

[**Report broken links or any other problems on this page.**](http://help.umuc.edu/)  
  
[**Copyright © by University of Maryland University College.**](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/common/copyright.html)